Geometry based analysis of 3R serial robots

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Abstract Cuspidal robots can travel from one inverse kinematic solution (IKS) to another without meeting a singularity. This property can be analyzed by understanding the inverse kinematic model (IKM) as well as the singularities in the joint space and in the workspace. In this article, we revisit the geometrical interpretation of the IKM with conics. The conditions of getting different conics and their implication on singularities are discussed and the observations regarding the nature of the conics are presented. Further, a sufficient condition for a 3R robot to be binary (i.e. with up to 2 IKS) as well as quaternary (i.e. with up to 4 IKS) is put forth by analyzing the geometrical interpretation of the IKM. The possibility to derive a necessary and sufficient condition is presented too.

1 Introduction

Cuspidal robots can travel from one inverse kinematic solution (IKS) to another without encountering a singularity. For these robots, posture identification is very difficult [1], which makes the task of trajectory planning more challenging [2]. It is known that a 3R robot can have at most four IKS, and it is generally preferred to choose a robot geometry that maximizes the size of regions with four IKS. This allows an end-user to choose IKS from different regions to counter the collision issues in the workspace. Robots that have 4 IKS regions in their workspace are referred to as *quaternary* robots, while robots that have at most 2 IKS are referred to as *binary* robots [3]. It is im-

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portant to note that though quaternary robots have their advantages, they can be cuspidal too, while on the other hand binary robots cannot be cuspidal [1]. A particular class of 3R robots, *orthogonal* 3R robots, have been studied extensively. These robots have three mutually orthogonal joint axes. A D-H parameter based condition for an orthogonal 3R robot to be quaternary was provided by algebraic analysis in [3]. An extension of such an analysis to non-orthogonal 3R robots is more challenging, and no conditions for binary or quaternary non-orthogonal robots have been reported yet. Recently, the cuspidality of *generic* 3R robots was analyzed by using a geometric interpretation of the inverse kinematic model (IKM) [4]. This work studied the intersection of a conic with a unit circle and derived important observations regarding the existence of reduced aspects as well as the necessary condition for cuspidality in generic 3R robots. The presented work reports few more properties of the conic and its implication on the maximum number of IKS in the workspace. The classification presented provides a simple and clear geometric interpretation for the condition of binary and quaternary robots. This work can be extended to have a necessary and sufficient condition for a generic 3R robot to be quaternary. The following work is divided into three sections: Section 2 revisits the geometrical interpretation of the inverse kinematics of 3R robots and singularities. This section also discusses the classification of 3R robots based on the geometry of the IKM as well as discusses the implication of the same in joint space and workspace. Section 3 shows a case of binary and quaternary robot by analyzing the geometrical properties of the IKM. Section 4 concludes the work by discussing the implications of the contribution and addressing a few pointers to future work.

2 Inverse kinematic model

Let $\mathbf{x} = (x, y, z)$ be the vector of coordinates of the robot's end effector in the workspace $\mathcal{W} \subset \mathbb{R}^3$ at a configuration $\mathbf{q} = (\theta_1, \theta_2, \theta_3)$ in the joint space $\mathcal{J} = S^1 \times S^1 \times S^1$. The mapping between $\mathcal{J}$ and $\mathcal{W}$, denoted by $f : \mathcal{J} \to \mathcal{W}$, defines the direct kinematics: $\mathbf{x} = f(\mathbf{q}), \mathbf{x} \in \mathcal{W}, \mathbf{q} \in \mathcal{J}$. The elements in the preimage $f^{-1}(\mathbf{x})$ are the IKS of $\mathbf{x}$. A robot configuration associated with an IKS is called a *posture*.

Solving the inverse kinematics of 3R serial robots was first reported in [5] where it was noted that the solutions correspond to the intersection of a conic with a circle in c_3s_3-plane, where c_3 and s_3 denote $\cos\theta_3$ and $\sin\theta_3$, respectively. Using the classical D-H parameters to describe the geometry of the robot (see [4]), the solution to the IKM is revisited briefly. Let, $R = \rho^2+z^2$, where $\rho^2 = x^2 + y^2 = g(\theta_2, \theta_3)$. The terms R and z can be written as

$$R = (F_1 \cos\theta_2 + F_2 \sin\theta_2)\, 2a_1 + F_3$$
$$z = (F_1 \sin\theta_2 - F_2 \cos\theta_2)\, \sin\alpha_1 + F_4$$

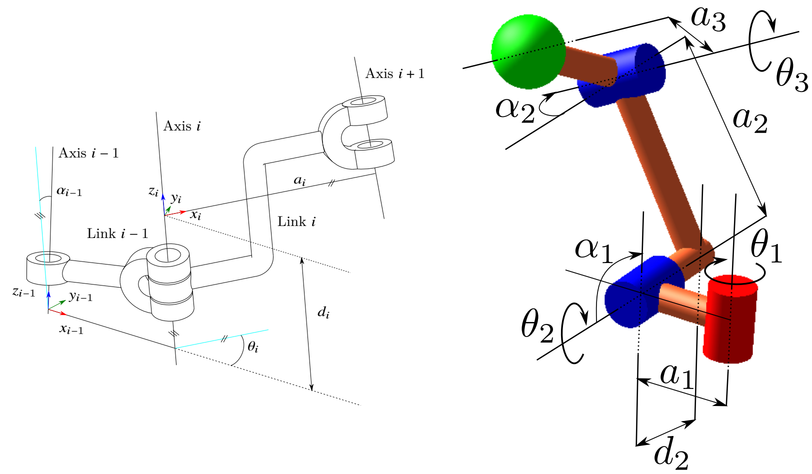

Fig. 1: D-H convention used and a schematic of a 3R serial robot

where $F_i = g_i(\theta_3)$, for $i = 1, .., 4$. Upon rearrangement, we obtain the general equation of a conic in $c_3 s_3$-plane as given in (1).

$$A_{xx}\, c_3^2 + 2A_{xy}\, c_3 s_3 + A_{yy}\, s_3^2 + 2B_x\, c_3 + 2B_y\, s_3 + C = 0 \tag{1}$$

The coefficients of the conic are skipped for brevity, but they are functions of the D-H parameters and of, (R, z) as shown in (2),

$$\begin{aligned} A_{xx}, A_{xy}, A_{yy} &= f_1(a_1, a_2, a_3, d_2, \alpha_1, \alpha_2) \\ B_x, B_y, C &= f_2(a_1, a_2, a_3, d_2, d_3, \alpha_1, \alpha_2, R, z) \end{aligned} \tag{2}$$

The inverse kinematic solutions are defined by the intersection points between the conic (1) and the unit circle $c_3^2 + s_3^2 = 1$ in $c_3 s_3$-plane. This conic can be a hyperbola, parabola or an ellipse depending on the D-H parameters. An example of each one is shown in Fig. 2.

Performing the tangent half-angle substitution, $t = \tan \frac{\theta_3}{2}$, we get a quartic inverse kinematic polynomial $M(t) = at^4 + bt^3 + ct^2 + dt + e$ similar to the one mentioned in [6]. The coefficients of $M(t)$ are functions of the D-H parameters and of R and z. The solutions to the polynomial equation, $M(t) = 0$, are the intersection points between the conic and the circle and are labeled as $\mathbf{m}_\psi$, where $\psi \in \{i, j, k, l\}$ in the $c_3 s_3$-plane.

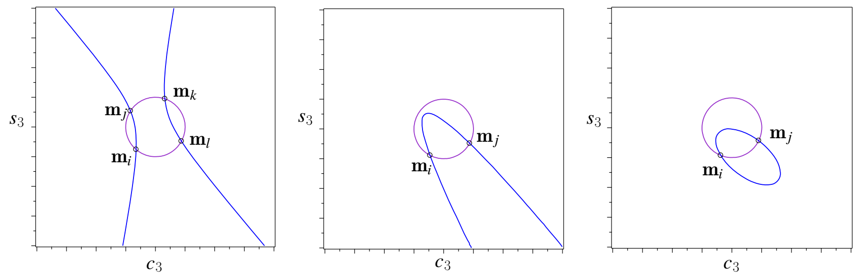

Fig. 2: The conic and unit circle in c_3s_3-plane for robots with different α parameters and at different points.
Robot parameters (left): $d = [0, 1, 0]$, $a = [1, 2, \frac{3}{2}]$, $\alpha = [\frac{\pi}{2}, \frac{\pi}{6}, 0]$, $(\rho, z) = (2.46, 0.15)$
(center): $d = [0, 1, 0]$, $a = [1, 2, \frac{3}{2}]$, $\alpha = [\frac{\pi}{3}, \frac{\pi}{2}, 0]$, $(\rho, z) = (2.33, -0.26)$
(right): $d = [0, 1, 0]$, $a = [1, 2, \frac{3}{2}]$, $\alpha = [\frac{\pi}{6}, \frac{\pi}{2}, 0]$, $(\rho, z) = (2.4, 0.6)$.

2.1 Singularities

It has been reported in [6] that the singularities in the workspace correspond to all the points where the root multiplicity of $M(t) = 0$ is greater than or equal to 2. Their geometric interpretation in the conic is two intersection points, $\mathbf{m}_i, \mathbf{m}_j$, merging together at a tangent point between the conic and the circle.
The nonsingular change of posture in cuspidal robots relates to two intersection points interchanging their position, with at least one of them not meeting any other intersection point [4]. This interpretation is also helpful to understand why binary robots are compulsorily noncuspidal. It has been shown in [4] that the orientation of the conic remains constant. Thus, if we have only two intersection points, then they cannot interchange their position without meeting each other at a tangent point.

2.2 Degenerate conic

The nature of the conic depends on the sign of the determinant of $\mathbf{N}$, where $\mathbf{N}$ is the Hessian of the conic. The degeneracy of a conic is given by $\det(\mathbf{D}) = 0$, where $\mathbf{D}$ is the Hessian of the quadratic form. We know that the hyperbola ($\det(\mathbf{N}) < 0$) degenerates into two intersecting lines, while the ellipse ($\det(\mathbf{N}) > 0$) degenerates to a point. The degenerate case of a parabola is of particular interest as it degenerates to two parallel lines, and they can be distinct or coincident. The presented work discusses the case when a parabola degenerates to two coincident lines, resulting in two multiple roots. This case is important to analyze as this is a special degeneracy case, and it is expected to show a special property in the workspace too. Following are the conditions for a parabola to degenerate into two coincident lines:

$$\begin{cases} \det(\mathbf{N}) = 0 \\ \det(\mathbf{D}) = 0 \\ B_x^2 + B_y^2 - (A_{xx} + A_{yy})\, C = 0 \end{cases} \tag{3}$$

Solving $\det(\mathbf{N}) = 0$ for d_2 yields:

$$d_2 = \pm\frac{\sqrt{(a_1+a_2)\,(a_1-a_2)\,(sa_1-sa_2)\,(sa_1+sa_2)}}{sa_1\, sa_2} \tag{4}$$

We conclude that the parameters a_3 and d_3 do not play any role to define a parabola. Upon substituting any value from (4) into $\det(\mathbf{D}) = 0$ and solving for R, we obtain the same solution. Solving the last equation in (3) for z, the solutions R and z take the following form, provided that $ca_2 \neq 0$ and $sa_1 \neq sa_2$:

$$\begin{aligned} R &= \frac{f(z)}{sa_1^2 - sa_2^2} \\ z &= \frac{ca_1}{ca_2}\,(d_3\, sa_1\, sa_2 + d_2\, ca_2) \end{aligned} \tag{5}$$

From (5), it is interesting to note that a robot such that ($ca_2 \neq 0$, $sa_1 \neq sa_2$) and corresponding to a parabola will always have a point in the workspace (R, z) such that its geometric interpretation is a parabola degenerating into a coincident line. When $ca_2 = 0$ (resp. $sa_1 = sa_2$), z (resp. R) is indeterminate. A point satisfying (5) is a tangency point between two loci of critical values, as shown in Fig.3.

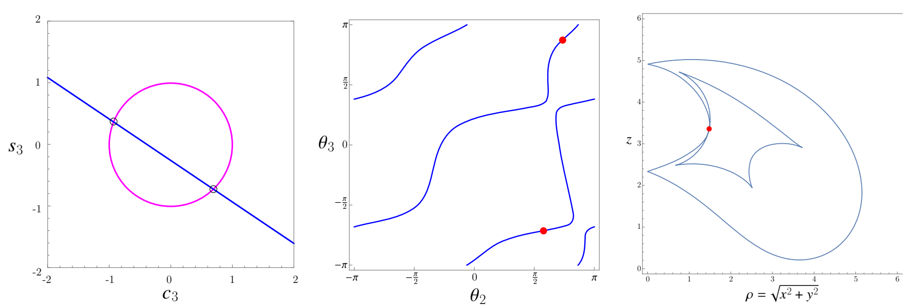

Fig. 3: Degenerate parabola case: joint space (center) and workspace (right). The point shown in red is associated with two coincident lines in the c_3s_3-plane (left).
Robot parameters: $d = [0, 2.828, 0.5], a = [1, 2, \frac{3}{2}], \alpha = [\frac{\pi}{6}, \frac{\pi}{3}, 0], (\rho, z) = (1.471, 3.315)$.

3 Special classes of robots

In this section, we present a sufficient condition for a 3R robot to be binary and quaternary, respectively. The motivation for the search of binary robots comes from the well-known property that two circles have at most two distinct intersections. Section 3.1 discusses the neighbourhood of such binary robots. We claim that the parameters corresponding to all the ellipses that are in a sufficiently small neighbourhood of the parameters corresponding to a circle, also result in binary robots. Section 3.2 discusses a condition for a hyperbola to compulsorily have four intersections with the unit circle in the c_3s_3-plane, thus resulting in a quaternary robot.

3.1 Binary robots

We now consider a robot such that its associated conic is an ellipse. We prove that if the ellipse is almost a circle, then many of these robots are binary. We consider a Lemma in Geometry whose proof is straightforward and thus not presented in the text:

Lemma 1 *Consider the unit circle S^1 and a number, $e \in (0, 1)$*

(i) There is an ellipse C with eccentricity e such that $\#C \cap S^1 = 4$
(ii) As $e \to 0$ the ellipses with eccentricity e with property (i) will have centers that approach the origin (center of S^1) and have minor and major semi-axes that approach length 1.

We now make an important remark that was not emphasized in [4]: given a robot, the eccentricity of its associated conic is fixed. Indeed, the eccentricity is only dependent on the entries of $\mathbf{N}$ (see [4, 5]), and these entries only depend on the D-H parameters of the robot and not on the position of the end-effector, as shown in (2). By combining this observation along with the lemma, we can prove the following theorem:

Theorem 1 *There are infinitely many binary robots whose associated conic is an ellipse (that is not a circle).*

Proof We claim that it suffices to have one binary generic robot with this condition. The associated ellipse for such a robot will never degenerate, and so the minor and major axes must achieve their (non-zero) minimum values. From previous Lemma, these lengths must lie in an interval $I \subset \mathbb{R}$ centered at 1 (radius of an ellipse) for the ellipse to intersect the unit circle four times and for a fixed sufficiently small eccentricity e. If a generic binary robot is given with an associated ellipse of eccentricity e, then the minimum major (or minor) axes (recall the axes' length now depends on the end-effector position) is outside I. This minimum value is not in the boundary of I and is continuously dependent on the D-H parameters. One parameter that does not affect the eccentricity but does affect the minimum major/minor axes' length is d_3. So we may perturb d_3 within a small interval of d_3 of the given binary robot and still obtain a binary robot.

To conclude the proof, we give an example of a binary robot whose associated conic is an ellipse. The D-H parameters of this robot are:

$$a = [-\frac{1503}{1879}, -1, -1] \quad d = [0, 0, 3] \quad \alpha = [-2.21, \frac{-\pi}{2}, 0]$$

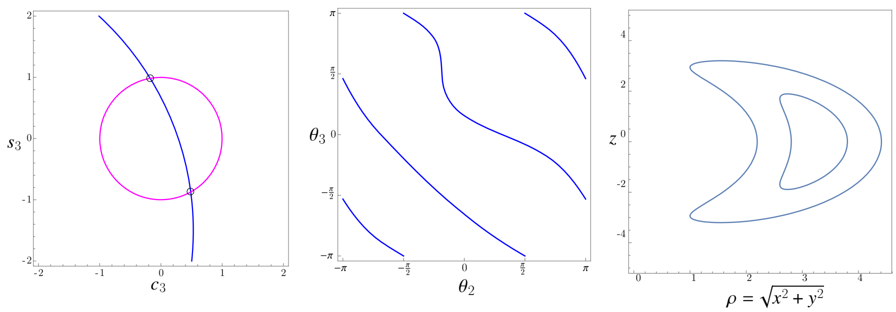

Fig. 4: An example of a binary robot's representation in $c_3s_3 - plane$, the joint space and the workspace.
Robot parameters: $d = [0, 0, 3]$, $a = [-\frac{1503}{1879}, -1, -1]$, $\alpha = [-2.21, \frac{-\pi}{2}, 0]$.

As the eccentricity of the associated ellipse approaches 0 (the ellipse becomes more like a circle) we will get more such binary robots. However, we will always be able to find a robot that is quaternery for such an eccentricity (as long as it is not 0). This is a conjecture that we aim to prove in the future.

3.2 Quaternary robot

In this subsection, we discuss the case of a 3R robot such that the hyperbola degenerates and the center of the conic is inside the circle. It is straightforward to argue why such a robot is compulsorily quaternary. If the intersecting lines have their intersection point inside a circle, then each line will intersect the circle twice, thus yielding four intersection points in total. If c_x, c_y is the center of the conic in the c_3s_3-plane, then the sufficient condition for a quaternary robot is:

$$\begin{cases} \det(\mathbf{N}) < 0 \\ \det(\mathbf{Q}) = 0 \\ \sqrt{c_x^2 + c_y^2} < 1 \end{cases} \tag{6}$$

To illustrate the simplicity of the derivation of the sufficient condition, we present a case of orthogonal 3R robots (see an example in 5).

$$\begin{aligned} \det(\mathbf{N}) &= -\frac{a_3^4\, d_2^2}{a_1^2} \\ \det(\mathbf{Q}) &= \frac{a_3^4\, d_2^2\,(d_3^2 - z^2)}{a_1^2} \\ c_x^2 + c_y^2 &= \frac{aR^2 + bR + c}{4\, d_2^2\, a_3^2} \end{aligned} \tag{7}$$

In (7), a and b are functions of the D-H parameters only and are not expressed fully for brevity. It is clear from (7) that an orthogonal 3R robot always corresponds to a hyperbola in $c_3 s_3$-plane. The condition for degeneracy depends only on d_3 and z while the condition for the center of the conic to lie inside the circle is a quadratic in R. It is important to note that the degeneracy depends only on z and not on R and thus, for $z = d_3$, the conic is always degenerate. This property also leads to some interesting observations about the hyperbolas corresponding to an orthogonal 3R robot but are not discussed here for lack of space.

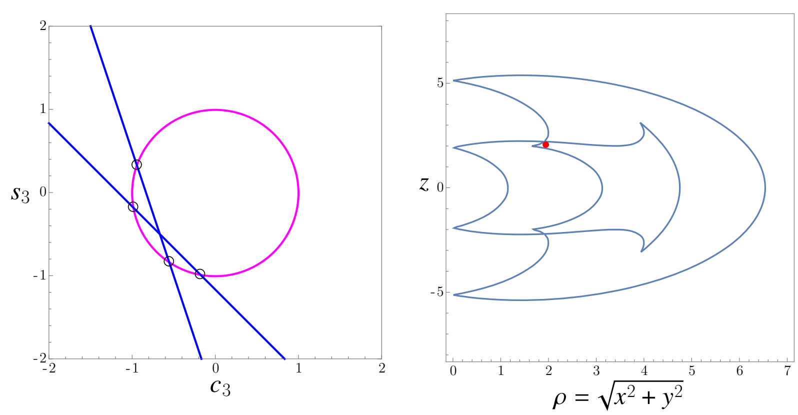

Fig. 5: An example of an orthogonal quaternary robot corresponding to an hyperbola. Robot parameters : $d = [0, 1, 2]$, $a = [1, 2, 3]$, $\alpha = [\frac{-\pi}{2}, \frac{\pi}{2}, 0]$, $(\rho, z) = (2, 2)$.

4 Conclusions and future work

In this work, we have revisited the geometric interpretation of the inverse kinematic model of 3R robots. The interpretation of singularities as well as the nonsingular change of posture have been briefly introduced. The special case of a parabola degenerating to two coincident lines is presented along with its interpretation in the workspace. The work also presented a sufficient condition for a 3R serial chain to be binary (ellipse case) (section 3.1) or quaternary (hyperbola case) (section 3.2) by using geometric observations. The advantages of the geometry based analysis is that the conditions for binary or quaternary robots can be extended to more generic cases of 3R robots without resorting to complex algebraic derivations. In future work, we aim to present a necessary and sufficient condition for a generic 3R robot to be binary or quaternary. This will allow the designer to include the condition while optimizing for a workspace with 4 IKS. We will also prove a few conjectures on binary and quaternary robots whose associated conic is an ellipse.

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